

Spectroscopy of a canonically quantized horizon

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Summary: Deviations from Hawking's thermal black hole spectrum, observable for macroscopic black holes, are derived from a model of a quantum horizon in loop quantum gravity. These arise from additional area eigenstates present in quantum surfaces excluded by the classical isolated horizon boundary conditions. The complete spectrum of area unexpectedly exhibits evenly spaced symmetry. This leads to an enhancement of some spectral lines on top of the thermal spectrum. This can imprint characteristic features into the spectra of black hole systems. It most notably gives the signature of quantum gravity observability in radiation from primordial black holes, and makes it possible to test loop quantum gravity with black holes well above Planck scale.

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I. INTRODUCTION

Most astrophysicists agree that black holes exist and radiate. So far three types of black hole radiations have been investigated: (i) the Hawking radiation, (ii) the gravitational radiation, and (iii) the X-ray emission from the infalling materials into a black hole. In this note, the quantum geometry of the horizon is, under certain assumption, shown to imply revision of the first type of black hole radiation.

The Hawking radiation is known semi-classically to be continuous. However, the Hawking quanta of energy are not able to hover at a fixed distance from the horizon since the geometry of the horizon has to fluctuate, once quantum gravitational effects are included. Thus, one suspects a modification of the radiation when quantum geometrical effects are properly taken into account. Any transition between two horizon area states can affect the radiation pattern of the black hole. The quantum fluctuations of horizon may either modify, alter or even obviate the semiclassical spectrum, [1, 2].

Bekenstein and Mukhanov in [3] studied a simple model of the quantum gravity of the horizon in which area is equally spaced. They found no continuous thermal spectrum but instead black holes radiate into discrete frequencies. The natural width of the spectrum lines turns out to be smaller than the energy gap between two consecutive lines. Thus, their simple model predicts a falsifiable discrete pattern of *equidistant* lines which are unblended. This result is not completely in contradiction with Hawking prediction of an effectively continuous thermal spectrum of black hole using semiclassical method, since the discrete line intensities are enveloped in Hawking radiation intensity pattern.

More recently, it has been possible to study the quantum geometry of horizons using precise method in loop quantum gravity. In this non-perturbative canonical approach, the quantum geometry is determined by geometrical observable operators. Canonical quantization of geometry supports the discreteness of quantum area.¹ This theory does not reproduce equally spaced area, instead the quanta become denser in larger values, [6, 7]. Having defined a black hole horizon as an internal *boundary* of space [8], only a subset of area eigenvalues contribute to identifying the horizon area. In fact, this subset contains the area associated to the edges puncturing the boundary. This subset is not evenly spaced and it turns out that the area fluctuations of such a horizon do not imprint quantum gravitational characteristics on black hole radiation, [9].

Nonetheless, restricting the quanta of horizon area to the subset of punctures is based on a non-trivial gauge-fixing of the horizon degrees of freedom. This is sufficient for the purpose of black hole entropy calculation since it results to the residence of a finite number of degrees of freedom on the horizon, independently from the bulk.

¹ A summary of emergent aspects of non-stringy quantum gravity theories can be found in [4], [5], and the references therein.

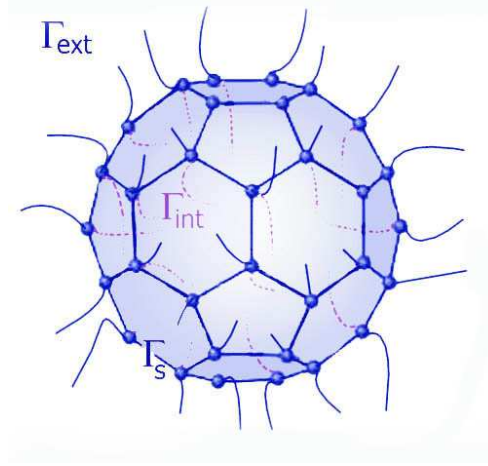


FIG. 1: A quantized black hole

Such a quantization, while is too restrictive, leaves some physical ambiguities. For instance, in classical general relativity spacetime metric field does not *end* at a black hole horizon, instead it extends through the black hole. In fact, a quantum black hole in a space manifold, instead of being the reason for termination of quantum space, partitions it into three subgraphs: 1) the partition that reside outside of horizon, Γ_{ext} , 2) the partition that reside inside of the horizon, Γ_{int} , and 3) the partition that lies on the horizon 2-surface, Γ_s . On the horizon surface some vertices and completely tangential edges reside. The spin network states associated to a partition that consists of the vertices lying on the horizon are called *horizon spin network states*. These states determine the whole quantum geometry of the underlying horizon, Fig (1). Under some simplifications, the spin network state associated to a spherical symmetric structure has been worked out in [12]. The quanta of such a horizon area is chosen from the complete spectrum. It reproduces the Bekenstein-Hawking entropy [10]. Moreover, in this note it is shown that such a black hole exhibits unexpectedly a macroscopic effect in the black hole radiation.

The aim of this note is two fold:

1. Firstly, in Part (I) an unexpected symmetry in the complete spectrum of area is described. In fact, this spectrum can be decomposed into a several *evenly* spaced sets, each with individual gap between levels. This leads to a reduced formula of area eigenvalues. In $SU(2)$ version of loop quantum gravity the gaps scale as the square roots of 'square-free' numbers. In $SO(3)$ version, they are the square

roots of the discriminants of all possible quadratic positive definite forms.

2. Secondly, in Part (II) it is discussed that having applied the complete spectrum of area, a black hole radiates quantum mechanically a continuous spectrum. But the existence of the symmetry within the area spectrum results to a phenomenon called the *quantum amplification effect*. This generates several distinct bright lines in radiance spectrum. It gives the signature of quantum gravity observability in radiation from primordial black holes. Moreover, it challenges the isolated horizon picture conjecture, while makes it possible to test loop quantum gravity with black hole radiation well above Planck scale.

Before these, some of the attempts to discovering the signature of quantum gravity in a black hole radiation are reviewed.

II. SOME THEORIES

Firstly, a model of quantum gravity that predicts macroscopic effects on black hole radiation is reviewed. Afterwards the attempts within loop quantum gravity are illustrated.

A. A quantum geometry and black hole radiation

A sector of spacetime may collapse and settles down to a stationary state in which the zeroth law of black hole mechanics is satisfied; the surface gravity is constant over the event horizon of the sector. The sector is called black hole. The ADM mass of a neutral non-rotating black hole non-trivially depends on the black hole horizon area. This is the first thermodynamic law of black holes,

$$A = \frac{16\pi G^2}{c^4} M^2. \quad (1)$$

Steven Hawking uncovered that quantum field theory in black hole curved spacetime leads to particle creation effect at the horizon, thus black hole radiates. The original derivations of this radiation was made of particle propagating into the black hole, the radiation is independent of the notion of particle, [14]. The sum of the black hole entropy plus the matter entropy outside the black hole never decreases,

$S_{\text{outside}} + S_{\text{black Hole}} \geq 0$. This is the generalized second thermodynamic law of black hole. This law holds even during quantum evaporation of the black hole via Hawking radiation, when a negative energy flux across the horizon decreases of area. Although, the way a black hole loses mass during the thermal radiation implicitly must involve quantum gravitational assumptions.

Jacob Bekenstein and Venceslav Mukhanov postulated a rough theory of quantum gravity in which the horizon area of a black hole is quantized in uniformly spaced tiny fractions of the Planck length scale,

$$A = \alpha n \ell_P^2, \quad (2)$$

where n is a natural number, and ℓ_P is Planck length, $\sqrt{\hbar G/c^3} \sim 1.6 \times 10^{-35} \text{m}$, which is drastically small.

Semiclassically, the discreteness of the quantum values of a horizon area leads to the discreteness of black hole mass. If a black hole is defined as a quantum system in thermodynamical equilibrium, the radiation is analogous to quantum mechanical instability that leads to quantum decays. Having the energy levels of a non-rotating neutral black hole, the transitions between neighboring energy levels causes quantum decay. A discrete mass spectrum implies the discreteness of mass emissions. From (2) and (1) the quanta of energy are

$$\delta M = \frac{\alpha \delta n}{32\pi M} M_P, \quad (3)$$

where M_P is the Planck mass, $\sqrt{\hbar c/G} \sim 2.2 \times 10^{-8} \text{ kg}$.

Under the assumption that the black hole mass is not changed during the quantum emissions, $\delta M \ll M$, and by the use of (3), the frequencies of emissive quanta turn out to be integers multiplied by a minimal frequency. The minimum frequency is called *the fundamental frequency* ϖ ,

$$\varpi = \frac{\alpha c^3}{32\pi G M}. \quad (4)$$

Other emissive frequencies are *harmonics* ω_n , which are proportional to this fundamental frequency by an integer n , $\omega_n = n\varpi$.

Under the assumption of uniform matrix elements of quantum transitions between near levels, the intensities of the spectral lines were worked out in [3]. The outcome

turn out to be enveloped by the Hawking radiation intensity, whilst the allowed frequencies are discrete and equidistant. Moreover, it turns out that the thermal broadening of the lines are smaller than the gap between any two consecutive harmonics.

From the model three major conclusion come about, (i) there should be no lines with wavelength of the order of the black hole size or larger, (ii) the black hole radiance spectrum must be clearly discrete and the lines do not overlap, (iii) the radiance pattern is a *uniformly spaced* discrete lines.

Nonetheless, there has not been any justification for this evenly spaced area from within the very quantum gravitational theories. In the next sections we consider a version of quantum gravity whose roots are within the so-called loop quantum gravity.

B. A quantum geometry and isolated horizon radiation

The first suggestion to describe a black hole as a 2-surface boundary of space manifold in loop quantum gravity was proposed by Krill Krasnov in [15]. Carlo Rovelli based on the picture discussed the black hole entropy in [16]. Afterwards, by the developments of the isolated horizon theory the bounded sector was more precisely defined in a series of works, [8].

A black hole is a classical concept and its definition is highly non-local, because one has to know the information of the entire spacetime manifold $(\Sigma, g_{\mu\nu})$, in order to find the entire causal past of the future null infinity. A black hole is a sector of manifold that does not intersect with the entire past of the future null infinity. However, this definition is not well-suited for the purpose of identifying a black hole region in a non-perturbative canonical quantized space. In fact, a more local criteria must be installed on such a theory.

A classical isolated horizon is defined by a set of boundary conditions of a sector of spacetime Δ , which mimics the essential local structure of a static event horizon. Assuming the black hole sector to be $S^2 \times \mathbb{R}$, these boundary conditions are necessary to verify the black hole thermodynamic laws from the sector: 1) the Einstein equations hold at the sector, 2) the sector is null, 3) the sector is equipped with a preferred foliation by 2-spheres transverse to its null normal l^a ; the second null normal to S_Δ is n^a with $l^a n_a = -1$, 4) the sector is non-rotating, 5) l^a is twist-, shear-, and expansion-free geodesic; n^a is twist- and shear-free with negative expansion $\theta_{(n)}$, 6) $\theta_{(n)}$ is constant

over each foliated shell S_Δ , 7) the flux densities of electric and magnetic fields are uniform through each S_Δ .

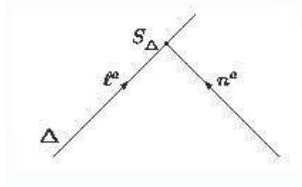


FIG. 2: The black hole sector Δ , its two null normals and its preferred foliation

S_Δ .

On the other hand, it is known that general relativity can be written in terms of gauge field. For this aim, a trivial $SU(2)$ bundle is assumed over the space 3-manifold. For each positive real number γ , a phase space ${}^\gamma\Pi$ is assumed to exist. This phase space consists of the configuration fields, the connection fields ${}^\gamma A_a^i$ (1-form), and the canonical momenta, the fields ${}^\gamma \tilde{\Sigma}_{ab\ i}$ of density weight one (2-form). $i = 1, 2, 3$ the gauge degrees of freedom and $a = 1, 2, 3$ the spatial degrees of freedom. The curvature of the connection field is a 2-form field ${}^\gamma F_{ab}^i$. The Einstein equations for any γ is verified.

The so-called ‘triad fields’ ${}^\gamma E_i^a$ are defined via the momentum fields $\tilde{E}_i^a := \gamma \varepsilon^{abc} {}^\gamma \tilde{\Sigma}_{bc\ i}$, where ε^{abc} is Levi-Civita ε^{abc} of density weight one. From the triads the 3-metric variables q^{ab} are defined, ${}^\gamma \tilde{E}_i^a {}^\gamma \tilde{E}^{bi} = q^{ab}$. Also, the triad fields define area of a 2-surface. Since the area of a 2-surface is $\int_S \sqrt{q} d^2x$, given the relation between the momenta and the 3-metric, the area can be redefined as $\int_S d^2x \sqrt{{}^\gamma E_i^a n_a {}^\gamma E_i^b n_b}$, a functional of momenta, where n_a is the normal to the surface.

In this language, a neutral stationary black hole in the manifold is the problem of adding a boundary with special boundary condition to the theory. The black hole sector is Δ where it is foliated by $S^2 \times \mathbb{R}$. The boundary of the sector, S_Δ , must satisfy the above mentioned conditions of an isolated horizon. In the gauge language of gravity, there is a way to define two null vectors of desired properties $l^1 n_a = -1$, $l^a l_a = n^a n_a = 0$ by the use of triad momentum conjugate fields ${}^\gamma \tilde{\Sigma}_{ab\ i}$. Having the two null vectors the following conditions must be imposed further in order to make a quasi-local black hole:

Area-fixing: the manifold momenta must admit a *fixed* value of area a on the shell.

Gauge-fixing: the pullback $\overleftarrow{\gamma A_a^i}$ of the bulk connection fields to the shell S_Δ are the $U(1)$ connection fields γW_a , up to a constant. For this aim, a $U(1)$ sub-bundle is selected at the shell S_Δ . By fixing a unit vector r^i at every point of the shell, the connection field on the sub-bundle will be $u(1)$ -valued γW_a .

Boundary condition: the pullback $\overleftarrow{\gamma \Sigma_{ab i}}$ of the bulk momenta to the shell S_Δ are completely determined by the curvature $\gamma F_{ab} = \partial_a \gamma W_b - \partial_b \gamma W_a$. The relation between these two turns to be $\gamma F_{ab} = -(2\pi\gamma/a) \overleftarrow{\gamma \Sigma_{ab i} r^i}$.

The field equations: at the sector the equations of motion hold.

The contribution of the boundary in the gravitational action is the addition of a $U(1)$ Chern-Simons action term of the gauge fields γW_a . Such an action is invariant under the following transformations: 1) $SU(2)$ gauge transformation, those reduce to $U(1)$ transformation on S_Δ and identity on the infinity, 2) spatial diffeomorphism, those reduce to tangent transformation on the shell and the identity at the infinity, 3) time evolution between the fixed horizon and infinity with lapse going to zero at the horizon and a constant at the infinity, 4) phase space transformation between different γ -sectors $\gamma \Pi$ and $\gamma' \Pi$.²

Such a classical horizon does not carry independent degrees of freedom due to the existence of the strong boundary condition. However, the quantum version is different.

To quantize the theory, a graph is embedded into the manifold and the connection fields are generalized into $su(2)$ -valued holonomies along the pathes of the graph. Two Hilbert spaces are obtained, the one of the bulk $\mathcal{H}_{\text{bulk}}$ and the one of the boundary $\mathcal{H}_{\text{boundary}}$. The boundary Hilbert space is defined on the Chern-Simons charged points, namely ‘punctures’ of the surface. The bulk Hilbert space has a basis by *spin networks* in the spatial 3-manifold with ‘loose ends’ at the charge points of the internal boundary.³

² In the quantum version, the quantum phase space $\gamma \Pi$ is unitarily *inequivalent* to the one of another quantum phase space $\gamma' \Pi$.

³ Quantization of a black hole has not been understood yet. There are several models for this purpose. Among them those are acceptable that do not make serious contradictions with the certain classical properties of a horizon. Let us consider non-perturbative context of quantum gravity. One of the recent model introduces a quantum black hole based on the action of ‘expansion operator’ on a the quantum state of a mixture of geometry and matter, [18]. There is another model based on

Consider a spin network state in the bulk Hilbert space. In this wave function, the edges of the spin network are labeled by the irreducible representation of the holonomies (the so-called ‘spins’), the vertices are intertwiners, the punctures by a vector $|m\rangle$ in the representation of the incident edge. If the spin of the incident edge to the puncture is j , there exist $2j+1$ different copies of puncture states, $m \in \{-j, -j+1, \dots, j-1, j\}$.

Notice that each puncture is a place where an edge ends at the surface and thus it carries the area eigenvalue corresponding to the edge.

The surface Hilbert space $\mathcal{H}_{\text{boundary}}$ contains $u(1)$ -valued connection fields. The geometry of the surface is flat except at the punctures, where there are conical singularities. All of different horizon wave functions corresponding to one edge of spin j produce the same horizon area, because the area eigenvalues only depend on j . Carlo Rovelli and Lee Smolin verified this area first by the use of loop operators in [6]. They found that the spectrum of area associated to an edge is ‘almost’ equidistant in large scales. The area of a puncture depends on the irreducible dimension of the puncturing edge. In fact, these eigenvalues of area were those were discovered first. An edge of spin j generates the area $a_j = 8\pi\gamma\ell_P^2\sqrt{j(j+1)}$ on the boundary. These eigenvalues depend only on one quantum number, j .

Having $2j+1$ different copies assigned to the same area, the horizon wave functions are degenerate, thus black hole gets non-zero entropy. The entropy is proportional to the surface area and since the surface is assumed to be of fixed area, the entropy of an isolated horizon meets the second thermodynamic law of black hole, it is non-decreasing. Therefore, the entropy is physical.

Later on Abhay Ashtekar and Jerzy Lewandowski derived the complete spectrum of area operator in [7]. The spectrum that Rovelli and Smolin have discovered was a subset of the complete spectrum of area. The complete spectrum is also discrete, although the eigenvalues approach to a continuum in large eigenvalues. This spectrum is described in section (II.C).

The gravitational fields about a black hole are not stable because they interact with ‘non-stationary’ matter fields. Only about such a shell, from the Einstein equation the decreasing of mass by ΔE corresponds to the decreasing of area by ΔA such that

causal dynamical triangulation. The causal dynamical triangulation is a non-perturbative quantum gravity analytically worked out in two dimensions in [20], statistically in [21], as well as numerically in higher dimensions. A black hole could be defined in this model, [19].

$\Delta A = 32\pi G^2 E \Delta E$. This correlation describes the transitions between two macroscopically stable states after mass perturbation. A quantum jumping down an area level corresponds to emission of one (or some) quantum of area. In both $SU(2)$ and $SO(3)$ versions of loop quantum gravity small values of spin j produces the quantum of area proportional to a number within the interval $[j, j + 1]$. At large j the area make it approximately proportional to j . Therefore, a transitions from a high level into a low level does not coincide with the transitions from a higher level into that high level. In other words, one-punctural decays produces an effectively continuous radiance spectrum at high frequencies.

However, the relation (3) since is a classical relation, does not guarantee the occurrence of only one-punctural transmissions. A quantum black hole may also radiate a multi-punctural decay in its low damped quasinormal modes. A multi-punctural decay is an emission in which a set of punctures simultaneously undergo area shrinking in one go and produce one quantum of energy. For instance, consider a black hole made of three patches of area, two of which correspond to punctures of spin $1/2$ and the third one to a spin 1 . The overall horizon area is $A_1 = 8\pi\gamma(\sqrt{2} + \sqrt{3})\ell_P^2$. This black hole may decay into a geometrical configuration with two punctures of spin 1 . In this case the horizon area is shrunk into $A_2 = 8\pi\gamma(2\sqrt{2})\ell_P^2$. According to (3) the emitted energy is proportional to $8\pi\gamma(\sqrt{3} - \sqrt{2})\ell_P^2$ by a constant, which is even smaller than the minimal single-punctural decay ($8\pi\gamma\sqrt{2}\ell_P^2$). Such a typical multi-punctural emissions can take almost any value and fill the continuous spectrum in all ranges of energy.

Since the puncture quantum of area is not uniformly spaced, the area fluctuations produces a continuous spectrum o emissive frequencies. While such a prediction satisfies the Hawking pattern of radiation, since the populations of all frequencies are uniform, there is no notable quantization effect in the black hole radiation, [9]. In the next section, a different picture of black hole is reviewed and its radiance pattern is analyzed.⁴

⁴ Beside these two possible pictures of a horizon in loop quantum geometry, there exists also a third one that was proposed by Livine and Terno in [17].

C. A quantum geometry and spin network horizon radiation

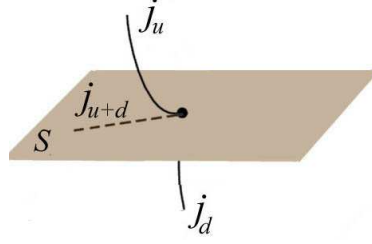
In this section a new picture of a black hole is explained and the quantum effects on its radiation is described in the rest of the note.

In brief, deriving the entropy of an isolated horizon depends on fixing a gauge of the connections fields. More precisely, in the presence of such a classical boundary-like horizon in the underlying manifold, the $su(2)$ -valued connection fields of the bulk are gauge-fixed into $u(1)$ -valued connections on the boundary and thus the punctures take additional degree of freedom independent from those of the bulk. This assumption is too restrictive. In such a quantum surface many quanta of horizon area are excluded by the classical isolated horizon boundary conditions. However, considering the complete spectrum of area eigenvalues as the possible horizon area, provides the same entropy that is expected on black hole horizon, while it gives a different picture of a black hole, a more quantum picture.

There have been some attempts to define a black hole as a partition of spin network state. For instance, Martin Bojowald in [12] tried to see a spherical symmetrical black hole state as a spin network state. In this picture, a black hole horizon is defined by studying the properties of an infalling spin network states associated to a 2-surface through the black hole. In this picture, instead of considering the evolution of the underlying surface and quantizing the manifold afterwards, the quantum state evolves itself independently. In fact, the behavior of the quantum surface in time can narrow the definition of appropriate dynamics of black hole.

Let us consider a quantum surface associated to a surface S in a 3-manifold Σ . This surface divides the manifold into two disjoint open sets Σ_{up} and Σ_{down} such that $\Sigma = \Sigma_{\text{up}} \cup S \cup \Sigma_{\text{down}}$ with $\Sigma_{\text{up}} \cap \Sigma_{\text{down}} = \emptyset$, Figure (3). Thus, the imbedding graph Γ in the presence of underlying 2-surface S is split into three subgraphs: (i) Γ_{up} , which is completely in one side of S in the 3-manifold, (ii) Γ_{down} , which is completely in the other side of S , and (iii) Γ_s , which lies on S . Γ_s consists of some residing vertices $\{v^\alpha\}$ on S as well as some tangential edge lying entirely on the horizon surface S .

Consider a typical spin network state corresponding to a residing vertex on an underlying surface S , i.e. the one in figure (3). This state intertwines the bulk edges of external and internal sub-graphs, and the edges of Γ_s . The set of all such spin networks produces a partition of spin network states called the *quantum surface states*. This state is isolated within the near-surface region. The quantum geometrical state of the surface is determined by these spin network states. A bulk edge relative to S

FIG. 3: Two incident edges at a vertex residing on S .

falls into three categories: either (a) it bends tangentially at the point at which the edge crosses the surface S , or (b) it intersects the surface at a point without bending at the surface, or (c) it lies completely tangential to the surface. The edges which are completely tangential to S , the so-called ‘analytical edges’, do not belong to the bulk edges, instead they belong to the quantum surface S .

In both the isolated horizon picture and the black hole spin network a quantum state is associated to the horizon. But in the latter one horizon is defined classically not quantum mechanically. The continuum surface undergoes evolution and a static Hilbert space is associated to the classically evolving surfaces at each time frame. In other words, the quantum state of horizon *follows* what the underlying surface *rules*. The quantization procedure of gauge-fixing prior to the quantization is not a trivial method of quantization. In fact, in the case of quantum spacetime such a quantization causes some ambiguities:

- The black hole sector is identified classically and remains exactly the same after quantization, without considering any uncertainty on black hole radius and its intrinsic geometry.
- The spin networks end at the black hole horizon, which contradicts with the classical definition of black hole. In classical general relativity, the metric fields extend through the horizon.
- No tunneling effect is allowed throughout the horizon.

To overcome the problems, one can treat the quantum horizon as an evolving quantum surface which undergoes its quantum evolution. The evolution is only expected to verify the classical results only at the classical limits. Such a quantum black

hole is a partition of spin network, whose boundary determines its quantum horizon. However, it is not so easy to define a surface without reference to a background metric or other fields. One surface that *can* be defined in a background-independent manner is a black hole horizon. This is a property that distinguishes horizons from most other surfaces. The final state of this partition should not be influenced by the initial states of the rest of the rest of the world. The initial state of this partition should influence the final states of the rest of the rest of the world. moreover, the entropy associated to the vertices residing on the horizon remains fixed. Also it is expected that the quantum sector gets non-expanding volume, as well as horizon area. This make it possible to make this definition of black hole more realistic because it the black hole be less hidden from a quantum system closer to the horizon, [23].

What is the entropy of such a quantum black hole? Considering a typical spin network state like the one of the figure (3), the action of area operator on this state generates an area eigenvalue. It turns out that an area eigenvalue corresponds to a *finite* number of different eigenstates. In fact, $su(2)$ -valued quantum states of a surface of certain area are degenerate state. The degeneracy is such that the entropy of the surface is proportional to the surface area and in the case of black holes quantum surface the entropy verifies the Bekenstein-Hawking entropy, [10]. This entropy is not necessarily non-decreasing in the course of time, unless it is fixed by the defining the appropriate evolution of horizon quantum surface states, [25].⁵

The complete eigenvalues of area operator on a typical spin network state was first studied in [7] and a few months later the results were verified by the use of recoupling theory in [22]. In fact, the area of a spin network is the outcome of linking the two sides of the surface. Let the up and down edges of the vertex α get the spin $j_u^{(\alpha)}$ and $j_d^{(\alpha)}$, respectively. The two edges may bend tangentially at the underlying 2-surface at their joint intersecting vertex. The overall tangent vector induced from them on the surface take the spin $j_{u+d}^{(\alpha)}$. The spin j_{u+d} take discrete values and bounded to the

⁵ The question why a horizon carries physical entropy whilst a random surface does not, is subtle and still not understood fully. This is not only a property of canonical quantization of spacetime. For instance, in the causal dynamical triangulation, which is another non-perturbative approach to quantum gravity [24], for the purpose of obtaining a 1+1 global geometry (a 2-surface) by triangular building blocks, the two components of the blocks can be respected as up and down spins with respect to an external time field, [21]. If one coarse-grain a ‘spin’ by ignoring the interior of some randomly selected region of the surface, one will obtain an entropy-like number. However, this number will not have the properties we normally associate with entropy.

following values

$$j_{u+d}^{(\alpha)} \in \{j_u^{(\alpha)} + j_d^{(\alpha)}, j_u^{(\alpha)} + j_d^{(\alpha)} - 1, \dots, |j_u^{(\alpha)} - j_d^{(\alpha)}| + 1, |j_u^{(\alpha)} - j_d^{(\alpha)}|\}. \quad (5)$$

The action of area operator on a typical area state corresponding to incident edges at a residing vertex on S entangles the external and internal edge. Let us for simplicity define the color numbers corresponding to the three spins, $p := 2j_u^{(\alpha)}$, $q := 2j_d^{(\alpha)}$, and $r := 2j_{u+d}^{(\alpha)}$. The area squared operator acting on the trivalent state $\langle p, q, r |$ entangles two sides of the underlying surface (the shaded and unshaded sides),

$$\begin{aligned} \left\langle \begin{array}{c|c} p & r \\ \hline q & \end{array} \right| \hat{A}^2 = -b^2 & \left(p^2 \left\langle \begin{array}{c|c} 2 & p \\ \hline q & r \end{array} \right| + q^2 \left\langle \begin{array}{c|c} p & r \\ \hline 2 & q \end{array} \right| \right. \\ & \left. + 2pq \left\langle \begin{array}{c|c} p & r \\ \hline 2 & q \end{array} \right| \right) \end{aligned} \quad (6)$$

where $b := 8\pi\gamma\ell_P^2$. Using the reduction formulae of recoupling theory, the grasped states are identical to the original state,

$$\begin{aligned} \left\langle \begin{array}{c|c} 2 & p \\ \hline q & r \end{array} \right| &= -\frac{(p+2)}{2p} \left\langle \begin{array}{c|c} p & r \\ \hline q & \end{array} \right|, \\ \left\langle \begin{array}{c|c} p & r \\ \hline 2 & q \end{array} \right| &= \frac{-2p(p+2)-2q(q+2)+2r(r+2)}{8pq} \left\langle \begin{array}{c|c} p & r \\ \hline q & \end{array} \right|. \end{aligned} \quad (7)$$

Substituting (7) in (6) the squared area operator acting on $\langle p, q, r |$ turns out to become an eigenstate relation non-trivially. Thus, the trivalent area state $\langle p, q, r |$ is the eigenstate of the operator. Finally, the area eigenvalues corresponding to the cell α turns out to be $a^{(\alpha)} = (4\pi\gamma) m_{j_u, j_d, j_{u+d}}^{(\alpha)} \ell_P^2$, where

$$m_{j_u, j_d, j_{u+d}}^{(\alpha)} = \sqrt{2j_u^{(\alpha)}(j_u^{(\alpha)} + 1) + 2j_d^{(\alpha)}(j_d^{(\alpha)} + 1) - j_{u+d}^{(\alpha)}(j_{u+d}^{(\alpha)} + 1)}. \quad (8)$$

A Schwarzschild black hole horizon belongs to the class of surfaces that has no boundary, $\partial S = \emptyset$ and divide the 3-manifold Σ into two disjoint sets Σ_{internal} and Σ_{external} such that $\Sigma = \Sigma_{\text{internal}} \cup S \cup \Sigma_{\text{external}}$ with $\Sigma_{\text{internal}} \cap \Sigma_{\text{external}} = \emptyset$. Thus, the graph Γ in the presence of a black hole is split into three graphs Γ_{external} , Γ_{internal} , and

Γ_s . Notice that the corresponding states to a compact closed surface can only gauge transform into another compact closed state. Therefore, a *subspace* of gauge invariant states those correspond to the compact closed surfaces are allowed to gauge transform into each other. Thus, further restrictions are imposed on this class of quantum states, [7]. The quantum states of a compact closed underlying geometry yields to those that satisfy the two conditions on the side bulk edge spins: $\sum_{\alpha} j_u^{(\alpha)} \in \mathbb{Z}^+$ and $\sum_{\alpha} j_d^{(\alpha)} \in \mathbb{Z}^+$. However, due to the existence of sum in these conditions, the spin of the majority of bulk edges in the near-horizon region are left unconditional. In other words, the conditioned trivalent states among all ingredient states of whole surface state is one or a few.

The quantum surface that is associated to a black hole horizon semi-classically determines the quantum decays of energy from the black hole. This definition is only restricted to the case of black holes and does not hold in any random surface. In the rest of the note the spectroscopy of the decays is illustrated. Before it, in the next part, an important symmetry within the eigenvalues are area operator is uncovered.

Part I

Part I: The symmetry of area spectrum

In this part, by the use of number theory a significant property of the area eigenvalues is uncovered. Having known the complete spectrum of area symmetry (8) a reduced formula is written. As a consequence, the complete spectrum of area eigenvalues in both group $SU(2)$ and $SO(3)$ representations can be split into the mixtures of equidistant numbers. This lead to the quantum amplification effect, which is described in next part.

III. $SO(3)$ AREA AND SQUARE-FREE NUMBERS

In $SO(3)$ group representation, the spins are positive integers. Evaluating $\frac{1}{2} (m_{j_u, j_d, j_{u+d}})^2$, if all repetitions of numbers (degeneracies) are identified, the whole *Natural* numbers are reproduced. This is proved in the Appendix (A).

As an immediate consequence, there exists a irreducible formula for the eigenvalues of area which depends only on one integer number. The irreducible formula of the complete area eigenvalues is

$$a_n = 4\pi\gamma\ell_P^2\chi\sqrt{n}, \quad (9)$$

where $\chi = \sqrt{2}$ and $n \in \mathbb{N}$.

In fact the eigenvalues of area operator in the original formula (8) that depends on three variables j_u, j_d, j_{u+d} is a reducible representation of the set. If degeneracies are identified the irreducible formula (9) appears.

Any integer is the multiplication of a ‘square-free’ number and a square number.⁶ By definition, an integer is said to be square-free, if its prime decomposition contains no repeated factors. For example, 30 is square free since its prime decomposition $2 \times 3 \times 5$ contains no repeated factors. Consider the natural number $2^5 \times 3^8$. This number can be rewritten in the form $(2) \times (2^2 \times 3^4)^2$. The first part is a square-free number and the second one is a square number.

Having this decomposition of natural numbers, consider the sequence of numbers containing the same square-free factor multiplied by all squared numbers. For example the sequence $\{3, 12, 27, 48, 75, 108, 147, \dots\}$, which is in fact $3 \times n^2$ for $n \in \mathbb{N}$. This sequence can be written in the form $3 \times \{1, 4, 9, 16, 25, 36, 49, \dots\}$, or briefly $3\mathbb{N}^2$. Such a sequence of numbers is called a squared set. The square-free number 3 based on which the sequence $3\mathbb{N}^2$ is produced is the *representative* of the squared set $3\mathbb{N}^2$. We indicate the representative with the symbol ζ and its corresponding square generation with $\zeta\mathbb{N}^2$.

Obviously, taking square root from the elements of a squared generation, say $\zeta\mathbb{N}^2$, an equidistant sequence of numbers is produced, $\sqrt{\zeta}\mathbb{N}$. This evenly spaced set of numbers is called a ‘*generation*.’ In fact, by doing this we decompose the set of numbers \sqrt{n} into $\sqrt{\zeta}m$ for integer n and m and square-free ζ . Consequently, the formula (8) is performed into the following reducible but important form:

$$a_n(\zeta) = (4\pi\gamma\ell_P^2\chi) n\sqrt{\zeta}, \quad (10)$$

where $\chi = \sqrt{2}$ $n \in \mathbb{N}$, and $\zeta \in \mathbb{A}$ for \mathbb{A} stands the set of square-free numbers. The list some of the square-free numbers are given in Table (I) of Appendix (A).

⁶ The proof is in Appendix (A).

What is special about this final formula is that it represents clearly that a generation with representative ζ gets evenly spaced area eigenvalues.

A curious reader is encouraged to read more details in the Appendix (A).

IV. $SU(2)$ AREA AND POSITIVE DEFINITE QUADRATIC FORMS

In $SU(2)$ group representation, evaluating $4(m_{j_u, j_d, j_{u+d}})^2$ from (8) produces the congruent numbers unto 0 or 3 mod 4. The proof is in the Appendix (B). These numbers are the page numbers of a book that is printed out only at the pages that come after each two leaves of sheets by face, 3, 4, 7, 8, 11, 12, 15, 16, 19, 20, etc. These numbers are also called *the Skew Amenable numbers*, [26].

Since the Skew Amenable number cannot be fitted into a formula with one variable. Instead, it can be fitted into the combination of these two sets: $(4\pi\gamma\ell_P^2\chi)\sqrt{4n}$ and $(4\pi\gamma\ell_P^2\chi)\sqrt{4n+3}$, where $\chi = 1/2$ and $n \in \mathbb{N}$. It can be proven that any skew amenable number b' can be written in terms of $b \times n^2$ for $n \in \mathbb{N}$. The numbers b are the elements of a subset of Skew Amenable numbers, the subset \mathbb{B} , that contains the discriminants of every positive definite quadratic forms.⁷

Henceforth, the complete spectrum of area eigenvalues $m_{j_u, j_d, j_{u+d}}$ is equivalent to the family of the generations $\{(\sqrt{\zeta}/2)\mathbb{N}\}$, where $\zeta \in B$. Area eigenvalues, instead of being determined by three quantum numbers j_u , j_d , and j_{u+d} , can be performed by two as

$$a_n(\zeta) = (4\pi\gamma\ell_P^2\chi) n\sqrt{\zeta}, \quad (11)$$

where $n \in \mathbb{N}$ and $\chi = 1/2$. The list of some of the discriminants is given in Table III) of Appendix (B).

A curious reader is encouraged to read more details in the Appendix (B).

V. CONCLUSION

Remarkably the area eigenvalues in a reduced form in both group representations $SU(2)$ and $SO(3)$ are performed into one formula. In the above two subsections it

⁷ For proofs refer to the Appendix (B).

was justified that the complete spectrum of area operator indeed can be specified by two indices n and ζ , instead of three indices j_u, j_d , and j_{u+d} ,

$$a_n(\zeta) = (4\pi\gamma\ell_P^2\chi) \ n\sqrt{\zeta}, \quad (12)$$

where $n \in \mathbb{N}$. In $SO(3)$ representation, $\zeta \in \mathbb{A}$ and $\chi = \sqrt{2}$. In $SU(2)$ representation, $\zeta \in \mathbb{B}$ and $\chi = 1/2$. χ is the group characteristic parameter and ζ is the generation representative. Therefore, in both group representations, the area eigenvalues exhibit equally spaced symmetry which make one of the original labels redundant.

Having defined the area eigenvalues, the following Lemmas can be easily investigated:

► **Lemma 1:** *Having two eigenvalues $a_1 \in \sqrt{\zeta_1}\mathbb{N}$, and $a_2 \in \sqrt{\zeta_2}\mathbb{N}$, where $\zeta_1 \neq \zeta_2$, for any choice of the eigenvalues in the corresponding generations these two eigenvalues are not equal, $a_1 \neq a_2$. ◀*

► **Lemma 2:** *Having two eigenvalues $a_1 \in \sqrt{\zeta_1}\mathbb{N}$, and $a_2 \in \sqrt{\zeta_2}\mathbb{N}$, where $\zeta_1, \zeta_2 \in \mathbb{A}$ (or \mathbb{B}) and $\zeta_1 \neq \zeta_2$, there is no eigenvalue in any generation that is equal to $a_1 \pm a_2$. ◀*

Part II

Part II: Radiation

In this part, based on the results of the Part (I), the spectroscopy of a quantum black hole is worked out.

Let us briefly overview the rest of this Part. The quantum fluctuation of the horizon area of a Schwarzschild black hole may occurs at the state associated to one or more than one of the horizon area cell. Since in the complete spectrum of area the gap between consecutive eigenvalues decreases in larger eigenvalues, effectively a continuous set of radiance frequencies are expected. Considering the result of Part (I), in which the complete spectrum of area is uncovered to exhibits evenly spacing symmetry if it is classified into some subsets (the so-called ‘generations’). Consider a transition from an upper area level, which belongs to the generation ζ_1 , into a lower area level, which belongs to the generation ζ_2 . While there is nothing special

with transition between two levels of two different generations, the radiance intensities of a set of frequencies which correspond to the transition within a generation ($\zeta_1 = \zeta_2 =: \zeta$) get highly amplified. The reason is that within a generation of quantum area a typical transition can occur from many different levels. For instance, a quantum leap of the scale of the double of the gap between a generation can be initiated from the third, fourth, fifth, up to the maximum levels. These quanta are all different copies of the same energy that a black hole may radiate. In fact, quantum amplification results into discrimination between the spectral line intensities. Such emissions are unblended and amenable to possible observation in primordial black holes.

Considering the symmetry of area each one of the generations justifies the equidistant ansatz (2) separately after replacing $\alpha = 4\pi\gamma\chi\sqrt{\zeta}$. The fundamental frequencies which are emitted by quantum leap inside a generation ζ is

$$\varpi(\zeta) = \frac{\gamma c^3}{8GM} \chi \sqrt{\zeta}. \quad (13)$$

Let us name $\omega_o := c^3/8GM$ the frequency scale factor and is of the order of $10^{16}/M_{kg}$ (eV). For instance the frequency scale corresponding to a black hole of mass $M \sim 10^{12}$ kg is of the order of 10 keV and thus the harmonics are of order $10\sqrt{\zeta}n$ keV, though these lines are not of the same intensities. This frequency if is associated to a primordial black hole it is subject to redshifting of the order of three order of magnitude. In fact, the intensity is suppressed as the gap between the levels of a generation grows. This because of the amount of quantum amplification a frequency may take. A precise work based on minimal number of natural assumptions is required to work out the intensities, which is introduced in the rest.

VI. QUANTUM AMPLIFICATION EFFECT

In general, transitions fall into two categories: (i) *the generational transitions*, quantum leaps from a level to a lower level of the same generation, and (ii) *the inter-generational transitions*, quantum leaps from a level into a lower level of different generation. The frequency corresponding to the first type of emissions is proportional to the fundamental frequency of the generation to which both the initial and final levels belong, $\omega_n := n\varpi(\zeta)$ for integer positive n . These frequencies are called *harmonic frequencies* of the generation ζ . Whatever frequency which is not of this type is of *non-harmonics*.

The strategy of determining the intensity of radiation is as follows. The intensity of an emissive frequency is defined by the amount of energy radiating at that frequency per unit time and area. The energy corresponding to a frequency is proportional to the average number of its emissive quanta. Firstly, it is assumed that the emissions occur in sequences. Accordingly, the probability of emissions of a typical sequence is determined. Having this, one can calculate the probability of the sequence that contains a number of the same frequencies. The average number of emissive quanta at different frequencies are determined. Thus, the intensity of frequencies are found. We calculate the intensity and the natural width of lines and the corresponding temperature to a black hole in this section. For the matter of clarifying the hidden assumption behind this strategy we give main axioms individually.

Axiom 1: Emissions occur in a sequential order.

This was first proposed by Ulrich Gerlach at the surface of a collapsing star in [27]. From a black hole, as a possible ultimate state of a collapsing star, a quantum of energy may be emitted between two classical stationary states. Describing the decay of the black hole during any interval of observer time Δt , a set of j individual decays are emitted in the sequence $\{\omega_1, \omega_2, \dots, \omega_n\}$, successively. The probability of a typical sequence is determined in this section.

In generational transitions, many copies of a harmonic frequency can be produced from different pairs. However, this is not the case for non-harmonics, because the irrational numbers $\sqrt{\zeta}$ that area eigenvalues are proportional to, cannot be decomposed into a sum of other irrational numbers. Accordingly, Lemma 2 approves that the difference $\Delta a = a(\zeta) - a'(\zeta')$ between two levels of different generations, $\zeta \neq \zeta'$, is 'unique' and cannot be produced by considering other pairs. Therefore, a non-harmonic transition is emitted only from *one* pair of levels.

On the other hand, from the classical relation between the horizon and the black hole mass (1), it is easy to verify $A(\text{m}^2) = 2.77 \times 10^{-53} M(\text{kg})^2$. The temperature of such a black hole is $T(\text{k}) = 1.23 \times 10^{23} / M(\text{kg})$. For instance, the horizon area corresponding to the black hole of mass 10^{12} kg is $2.77 \times 10^{-29} (\text{m}^2)$ and its temperature is $1.228 \times 10^{11} \text{ k}$. Such a horizon is 40 order of magnitude larger than the quanta of area! This gives the confidence that the number of levels that contribute to the radiation procedure is enormous.

This fact make the difference between harmonics and non-harmonics important.

Namely, the population of harmonics exceed the population of non-harmonics. This effect in quantum mechanics is called *Quantum Amplification Effect*. This effect has a strong root in the symmetry of area.

To determine how much the difference of the population is important and if it is visible a precise analysis is necessary. Let us start off the analysis with the probability of some decay in a sequential order.

A. The probability of time-ordered decays

The probability of one jump (no matter of what frequency) in the course of time Δt is shown by $P_{\Delta t}(1)$. Similarly, the probability of no jump is $P_{\Delta t}(0)$. During the time interval $2\Delta t$, the probability of no jump (the failure of decaying) is equal to the probability of the failure in each one of its two fragment of time intervals, $P_{2\Delta t}(0) = [P_{\Delta t}(0)]^2$. The general solution of this functional equation is $P_{\Delta t}(0) = \exp(-\Delta t/\tau)$, where τ is the survival timescale of the black hole from decaying.

We let the horizon to decay a sequence of frequencies successively. Using the same argument, the probability of one jump (of any frequency) in time interval $2\Delta t$ is $P_{2\Delta t}(1) = 2P_{\Delta t}(0)P_{\Delta t}(1)$. Therefore, $P_{\Delta t}(1) = (\Delta t/\tau') \exp(-\Delta t/\tau)$.

The probability of 2 jumps in the time interval $2\Delta t$ can be written as $P_{2\Delta t}(2) = 2P_{\Delta t}(0)P_{\Delta t}(2) + [P_{\Delta t}(1)]^2$. This formula can be extended to the probability of emission of ‘even’ number (j) quanta in the time interval $2\Delta t$,

$$P_{2\Delta t}(j) = 2 \sum_{i=0}^{j/2-1} P_{\Delta t}(i)P_{\Delta t}(j-i) + [P_{\Delta t}(j/2)]^2. \quad (14)$$

On the other hand, the probability of 3 jumps in the time interval $2\Delta t$ can be written as $P_{2\Delta t}(3) = 2P_{\Delta t}(0)P_{\Delta t}(3) + 2P_{\Delta t}(1)P_{\Delta t}(2)$. This formula can deduce to the probability of emission of ‘odd’ number (j) quanta in the time interval $2\Delta t$,

$$P_{2\Delta t}(j) = 2 \sum_{i=0}^{j-1/2} P_{\Delta t}(i)P_{\Delta t}(j-i). \quad (15)$$

Generating all the probabilities starting from $P_{\Delta t}(0)$ up to $P_{\Delta t}(j-1)$ consecutively from the recursive formula (14) and (15), one can generate the probability of j emissions as a function of j and Δt . The general solution for the probability of j decays

is

$$P_{\Delta t}(j) = \frac{1}{j!} \left(\frac{\Delta t}{\tau'} \right)^j \exp(-\Delta t/\tau), \quad (16)$$

in which by normalization $\tau' = \tau$.

B. The probability of a decay

Consider a sequence of radiance frequencies $\{\omega_1, \omega_2 \dots\}$. These frequencies might be harmonics or non-harmonics. For the purpose of determining the probability of this sequence let us begin with one jump ($j = 1$) in the course of time Δt .

Since the emissions are supposed to occur in time order, the probability of a sequence of decays is the product of conditional probability and the probability distribution of time ordering. Thus, the probability of one emission is $P_{\Delta t}(\{\omega\}) = P_{\Delta t}(\{\omega \mid 1\})P_{\Delta t}(j = 1)$. This probability is not difficult to be determined. Before this the second axiom is introduced.

Axiom 2: The entropy of a Schwarzschild black hole is dominantly $A/4\ell_P^2$.

Entropy is defined as the logarithm of the number of microstates of a macroscopic state. Since the macroscopic state of a Schwarzschild black hole is determined by one parameter, the horizon area, the entropy associated to a black hole of horizon area A is determined by the number of quantum states associated to such a horizon. This degeneracy, $g(A)$, is dominantly $g(A) = \exp(A/4\ell_P^2)$. The horizon area of a loop quantum black hole is made of N patches of area eigenvalues, $A = \sum_{i=1}^N a_i$. Therefore the black hole degeneracy is in fact $g(A) = \exp(\sum_{i=1}^N a_i/4\ell_P^2)$. On the other hand, the overall degeneracy g corresponding to a system that is made of N subsystems each with individual degeneracy g_i , is $g = \prod_{i=1}^N g_i$. Due to the Lemma 2, the contribution of each generation in the horizon area A cannot be replaced by the combination of area levels of the other generations. Therefore, the macroscopic horizon area is split into its ingredient area contribution of each generation, $g(A) = \prod_{\zeta} \exp(\sum_i a_i(\zeta)/4\ell_P^2)$. Therefore, the degeneracy associated to the generation τ is $g(\zeta) = \exp(\sum_i a_i(\zeta)/4\ell_P^2)$, where i indicates the levels of the generation that contribute in the horizon area. Since each generation is equidistant, all level that contribute in the horizon area from one generation sum into a level inside the same generation, say the level n . In other

words, $\sum_i a_i(\zeta) =: a_n(\zeta)$. Consequently, the degeneracy $g(\zeta)$ can be thought of being the degeneracy associated to the level n of the generation, $g(n; \zeta) = \exp(a_n(\zeta)/4\ell_P^2)$. By the use of equation (12), the degeneracy of a typical level $a_n(\zeta)$ is

$$g(n; \zeta) := e^{\pi\gamma\chi\sqrt{\zeta} n}. \quad (17)$$

where ζ is the representative number of the generation, n is the level of the frequency in the generation ladder.⁸

An area patch of level n of the generation ζ may decay into the level n' -th of the generation ζ' , where $\sqrt{\zeta}n > \sqrt{\zeta'}n'$. In this process, the degeneracy $g(n; \zeta)$ changes into $g(n'; \zeta')$. By the use of (17), the transition into a lower level changes the degeneracy by a factor of $\exp(-\pi\gamma\chi|\sqrt{\zeta}n - \sqrt{\zeta'}n'|)$ corresponding to the emission of the frequency $\omega = (\gamma\chi c^3/8GM)(\sqrt{\zeta}n - \sqrt{\zeta'}n')$.

The ‘population’ of a quantum of area is the number of different pairs of levels that produce it. This number can be normalized to one by the use of the maximum population number, N_o , which is in fact nothing but the number of level pairs that produce the fundamental frequency of the first generation (the generation whose corresponding gap between levels is minimal). Therefore, the *population weight* of the frequency ω is defined as $\rho(\omega) = N/N_o$, where N is the number of pairs that produce the frequency ω . It is also clear that the population weight of non-harmonics is $1/N_o$.

In our navigation for determining the probability of a specific frequency emission the third axiom is introduced:

*Axiom 3: The density matrix elements for quantum transitions
between near levels are uniform.*

Having assumed this, the probability of a jump is proportional to the change of degeneracy as well as the population weight of the frequency. Therefore, the conditional probability of a typical frequency ω by the use of (17) is

⁸ This degeneracy can also be determined exactly by considering the fact that a typical area level $a_n(\zeta)$ can be made of some smaller area patches (of the same generation) in some different configurations. Considering the degeneracy of each level, the exponentially growing degeneracy of a level is immediately reproduced, [23].

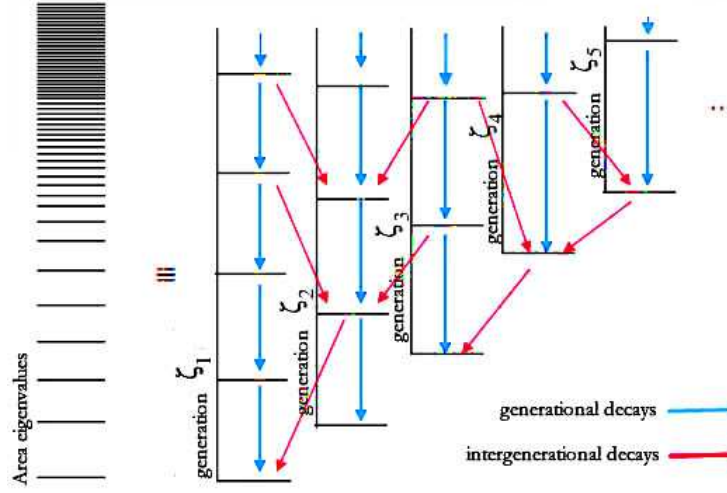


FIG. 4: A schematic diagram of generational emissions (the vertical black arrows) and inter-generational emissions (the slanted red arrows) for a few generations.

$$P_{\Delta t}(\{\omega\} | 1) = \frac{1}{C} \rho(\omega) e^{-\Lambda \omega}, \quad (18)$$

where

$$\Lambda := 8\pi GM/c^3. \quad (19)$$

The normalization relation of the probability determines C . It is defined to be $C := \sum_{\omega} \rho(\omega) e^{-\Lambda \omega}$.

Since the probability of a typical frequency ω is determined from $P_{\Delta t}(\{\omega\}) = P_{\Delta t}(\{\omega\} | 1) P_{\Delta t}(1)$. By the use of (16) and (18) for $j = 1$,

$$P_{\Delta t}(\{\omega\}) = \frac{\Delta t}{C\tau} e^{-\Delta t/\tau} \rho(\omega) e^{-\Lambda \omega} \quad (20)$$

In the case of generational decays, the decay condition $\sqrt{\zeta}n > \sqrt{\zeta'}n'$ is reduced into $n > n'$. The conditional probability of a typical frequency $\omega_m := m\varpi(\zeta)$ emission, where $m = n - n'$, by the use of (13) and (18) reads

$$P_{\Delta t}(\{\omega_m(\zeta)\} | 1) = \frac{1}{C} \rho(\zeta) q(\zeta)^{-m}, \quad (21)$$

where $q(\zeta) := e^{\Lambda\varpi(\zeta)}$ is independent of mass and dependent to the generational representative number ζ . In fact it is ⁹

$$q(\zeta) := e^{\pi\gamma\chi\sqrt{\zeta}}. \quad (22)$$

Given a black hole of horizon area A , it is discussed in section (VI) a generation with smaller gap between levels produces more copies of each one of its corresponding harmonic frequencies. Since the gap between the levels is $(4\pi\gamma\chi\sqrt{\zeta}) \ell_P^2$ by the use of (12) the number of levels below the horizon area is $N := A/(4\pi\gamma\chi\sqrt{\zeta})\ell_P^2$, which is a huge number (about 10^{40} levels). On the other hand, the number of m -level jumps down a ladder of total N levels is $N - m$. In a classical black hole N is extremely large, henceforth the population weight of the harmonics of frequency $\omega_m(\zeta)$ for $m \ll N$ is $\rho(\zeta) = N/N_o = \sqrt{\zeta_o/\zeta}$, where ζ_o is the generation with the minimal gap between levels.¹⁰ Dropping the constant coefficient $\sqrt{\zeta_o}$ from the definition, the population weight of small harmonics is

$$\rho(\zeta) := \frac{1}{\sqrt{\zeta}}. \quad (23)$$

Since the probability of the harmonic frequency $\omega_n(\zeta)$ is defined by $P_{\Delta t}(\{\omega_n(\zeta)\}) = P_{\Delta t}(\{\omega_n(\zeta)\} \mid 1) P_{\Delta t}(1)$. By the use of (21) and (16) for $j = 1$, one can write the probability of the harmonic frequency

$$P_{\Delta t}(\{\omega_n(\zeta)\}) = \frac{\Delta t}{C\tau} e^{-\Delta t/\tau} \rho(\zeta) q(\zeta)^{-n} \quad (24)$$

What is C ? In fact after a moment of analytical calculation the normalization coefficient C is found

$$C = \sum_{\text{all } \zeta} \frac{\rho(\zeta)}{q(\zeta) - 1} \quad (25)$$

It is easy to prove that C is a finite number of the order $O(1)$. A curious reader is encouraged to read the detail of the derivation of C and testing its finiteness in Appendix (D).

Next step is to generalized this probability for a sequence of j successive emissions of different frequencies.

⁹ Comparing (22) with the degeneracy of a level (17) shows the simple relation: $g(n; \zeta) = q(\zeta)^n$.

¹⁰ ζ_o in $SU(2)$ version is 3 and in $SO(3)$ version is 1.

C. The probability of a sequence of emissions

Following the Axiom 1, the generalized probability of a sequence of harmonics is $P_{\Delta t}(\{\omega_1, \omega_2, \dots, \omega_j\})$, where the frequencies can be harmonics or non-harmonics. Let us assume the time interval is made of S fragments of smaller time intervals, $\Delta t = S\epsilon$, where $S \gg j$ and each one of the j decays occurs in one fragment of time ϵ . There are $S!/j!(S-j)!$ number of ways for selecting j jumping intervals out of total S time intervals. This number of ways for the case of $S \gg j$ is approximated to $S^j/j!$. In the overall $j\epsilon$ moment intervals out of S ones, the black hole successfully decays and in the rest of time, $(S-j)\epsilon$, it fails to decay. The probability of j emissions is thus $(S^j/j!)P_\epsilon(0)^{S-j} \prod_{i=1}^j P_\epsilon(\{\omega_i\})$. Substituting $P_\epsilon(\{\omega_i\})$'s from (24), the probability is:

$$P_{\Delta t}(\{\omega_1, \omega_2, \dots, \omega_j\}) = \frac{1}{j!} \left(\frac{\Delta t}{C\tau} \right)^j e^{-\Delta t/\tau} \prod_{i=1}^j \rho(\zeta_i) e^{-\Lambda\omega(\zeta_i)}. \quad (26)$$

In the presence of r non-harmonics in a sequence of frequencies decreases the probability of the sequence by a factor of $(1/N_o)^r$, which is negligible for classic black holes. In fact, only the harmonics take a major contribution to determining the intensities.

Let us consider now a sequence of harmonic emissions of different generations, $\{\omega_{n_1}(\zeta_1), \omega_{n_2}(\zeta_2), \dots, \omega_{n_j}(\zeta_j)\}$. According to (21) and (26) the probability of the sequence is

$$P_{\Delta t}(\{\omega_{n_1}(\zeta_1), \omega_{n_2}(\zeta_2), \dots, \omega_{n_j}(\zeta_j)\}) = \frac{1}{j!} \left(\frac{\Delta t}{C\tau} \right)^j e^{-\Delta t/\tau} \prod_{i=1}^j \rho(\zeta_i) q(\zeta_i)^{-n_i}. \quad (27)$$

By the use of the probability of j decays in the course of time Δt from (16), the generalized conditional probability of a sequence of harmonics is found

$$P_{\Delta t}(\{\omega_{n_1}(\zeta_1), \omega_{n_2}(\zeta_2), \dots, \omega_{n_j}(\zeta_j)\} | j) = \left(\frac{1}{C} \right)^j \prod_{i=1}^j \rho(\zeta_i) q(\zeta_i)^{-n_i}. \quad (28)$$

This conditional probability turn out to be independent of time.

Since the intensity of a harmonic frequency depends on the average number of the emission in the course of time. This average number depends on the probability of k emissions of the emissive frequency in any sequence of dimension $j \geq k$.

D. The probability of k quanta of the same frequency

Let us assume that among the j emissions there exist k quanta of the frequency $\omega_{n_k}(\zeta_k)$ and the rest $j - k$ frequencies belong to other frequencies. Consider the j dimensional sequence $\{\omega_{n_1}(\zeta_1), \omega_{n_2}(\zeta_2), \dots, \omega_{n_k}(\zeta_k), \dots, \omega_{n_k}(\zeta_k), \dots, \omega_{n_j}(\zeta_j)\}$ in which there are k quanta of the same frequency $\omega_{n_k}(\zeta_k)$. If the black hole makes j decays such that k of them are of the same frequency $\omega_{n_k}(\zeta_k)$, (for $k \leq j$), there are $k!/j!(j-k)!$ ways to select these k quanta. The probability of each selection due to (28) is

$$\left(\frac{1}{C}\right)^j (\rho(\zeta_k) q(\zeta_k)^{-n_k})^k \prod_{i=1}^{j-k} \rho(\zeta_i) q(\zeta_i)^{-n_i}. \quad (29)$$

where in the product part of it the frequencies are any frequency except $\omega_{n_k}(\zeta_k)$.

For the purpose of determining the probability of k emissive quanta of the frequency $\omega_{n_k}(\zeta_k)$ in a j dimensional string included all possible accompanying frequencies, $P_{\Delta t}(k | \omega_{n_k}(\zeta_k), j)$, we should sum over the probabilities (29) for all possible frequencies associated to the accompanying frequencies, all frequency values except $\omega_{n_k}(\zeta_k)$. Since the non-harmonic emissions are continuous sum over non-harmonics is effectively evaluated by integral. We must consider different cases for the $j - k$ accompanying emissions: the case that none of the $j - k$ frequencies is non-harmonic, the case that only one of them is non-harmonic, etc.. Therefore the conditional prob-

ability is

$$\begin{aligned}
P_{\Delta t}(k|\omega_{n_k}(\zeta_k), j) = & \frac{j!}{k!(j-k)!} \left(\frac{1}{C}\right)^j (\rho(\zeta_k) q(\zeta_k)^{-n_k})^k \times \\
& \left[\prod_{i=1}^{j-k} \sum_{\text{all } \zeta} \sum_{\omega \neq \omega_{n_k}} \rho(\zeta_i) q(\zeta_i)^{-n_i} \right. \\
& + \prod_{i=1}^{j-k-1} \sum_{\text{all } \zeta} \sum_{\omega \neq \omega_{n_k}} \rho(\zeta_i) q(\zeta_i)^{-n_i} \left(\int_{0, \omega \neq \omega_k}^{\infty} \rho e^{-\Lambda \omega} \Lambda d\omega \right) \\
& + \prod_{i=1}^{j-k-2} \sum_{\text{all } \zeta} \sum_{\omega \neq \omega_{n_k}} \rho(\zeta_i) q(\zeta_i)^{-n_i} \left(\int_{0, \omega \neq \omega_k}^{\infty} \rho e^{-\Lambda \omega} \Lambda d\omega \right)^2 + \dots \\
& + \sum_{\text{all } \zeta} \sum_{\omega \neq \omega_{n_k}} \rho(\zeta_i) q(\zeta_i)^{-n_i} \left(\int_{0, \omega \neq \omega_k}^{\infty} \rho e^{-\Lambda \omega} \Lambda d\omega \right)^{j-k-1} \\
& \left. + \left(\int_{0, \omega \neq \omega_k}^{\infty} \rho e^{-\Lambda \omega} \Lambda d\omega \right)^{j-k} \right].
\end{aligned}$$

Substituting ρ , the contribution of a non-harmonic emission becomes $(1/N_o)(1 - e^{-\Lambda \omega_k})$. By the use of the equality (D2) the sum $\sum_{\text{all } \zeta} \sum_{n \neq n_k} \rho(\zeta) q(\zeta)^{-n}$ gives rise to $C - \rho(\zeta_k) q(\zeta_k)^{-n_k}$.

In the classical limit, $1/N_o \rightarrow 0$, inside the bracket all terms with the factor $1/N_o$ are higher order corrections to the probability. For the purpose of determining the intensity, it is sufficient to consider only the first order term. Effectively this probability is

$$\begin{aligned}
P_{\Delta t}(k | \omega_{n_k}(\zeta_k), j) = & \frac{j!}{k!(j-k)!} \left(\frac{1}{C}\right)^j (\rho(\zeta_k) q(\zeta_k)^{-n_k})^k \\
& \times (C - \rho(\zeta_k) q(\zeta_k)^{-n_k})^{j-k}.
\end{aligned} \tag{30}$$

We multiply this conditional probability by the absolute probability distribution $P_{\Delta t}(j)$ in equation (16) and sum over all $j \geq k$ in order to provide the probability of $P_{\Delta t}(k|n_k \varpi^{(\zeta)})$,

$$P_{\Delta t}(k | \omega_{n_k}(\zeta_k)) = \frac{1}{k!} \left(\rho(\zeta_k) q(\zeta_k)^{-n_k} \right)^k \left(C - \rho(\zeta_k) q(\zeta_k)^{-n_k} \right)^{-k} e^{-\Delta t/\tau} \\ \times \sum_{j \geq k}^{\infty} \frac{1}{(j-k)!} \left(\frac{\Delta t}{C\tau} \right)^j \left(C - \rho(\zeta_k) q(\zeta_k)^{-n_k} \right)^j$$

Applying the equality $\sum_{a=b}^{\infty} z^a / (a-b)! = z^b \exp(z)$ by replacing a, b with j, k respectively in the second line, the probability distribution of the emission k quanta of frequency $\omega_{n_k}(\zeta_k)$ is determined,

$$P_{\Delta t}(k | \omega_{n_k}(\zeta_k)) = \frac{1}{k!} (x_{n_k}(\zeta_k))^k e^{-x_{n_k}(\zeta_k)}, \quad (31)$$

where $x_n(\zeta) := (\Delta t / C\tau) \rho(\zeta) q(\zeta)^{-n}$. This probability turns out to be Poisson-like distribution.

VII. INTENSITY

By definition, the intensity of $\omega_{n_k}(\zeta_k)$ is the total energy that is emitted at this frequency and unit time per unit area. Since the emissions of diverse frequencies are independent, the total energy of a frequency is the average number of quanta emitted at that frequency times the energy of the frequency.

Using (31), this average number of quanta of this frequency is

$$\bar{k} = \sum_{k=1}^{\infty} k P_{\Delta t}(k | \omega_{n_k}(\zeta_k)) = \left(\frac{\Delta t}{C\tau} \right) \rho(\zeta_k) q(\zeta_k)^{-n_k}.$$

Since the mean value of the number of quanta emitted at a typical harmonic frequency $\omega_n(\zeta)$ is proportional to Δt as well as $\rho(\zeta) q(\zeta)^{-n}$, the intensity of a typical line $\omega_n(\zeta)$ is

$$I(\omega_n) = I_o \omega_n(\zeta) \rho(\zeta) e^{-\Lambda \omega_n(\zeta)}. \quad (32)$$

where I_o is a constant.

VIII. TEMPERATURE

In the thermal radiation from a black body the number of quanta in a frequency is distributed by a Poisson function, according to (31). To see the consistency of the Poisson distribution with the thermal distribution of a black body consider the radiation from a black body at a given temperature T . According to the definition of black body, the number of a frequency ω those are emitted from within the body is determined by the Boltzmann function, $P_T(\omega) = B \exp(-\hbar\omega/k_B T)$, where B is a normalization constant, $B = 1/\sum_{\omega} \exp(-\hbar\omega/k_B T)$, (similar to (21)) and k_B is Boltzman constant. The probability of k quanta emissions of a specific frequency ω_k in a j dimensional sequence of decays is $\binom{j}{k} P_T(\omega_k)^k \prod_{i \neq k} P_T(\omega_i)$. Summing over all accompanying frequency except ω_k , the conditional probability is

$$P_T(k|\omega_k, j) = \binom{j}{k} B^k \exp(-k\hbar\omega_k/k_B T) (1 - B \exp(-\hbar\omega_k/k_B T))^{j-k}. \quad (33)$$

Comparing this conditional probability and the one of (30), they are the same for a frequency ω_n if the coefficients of the two exponents are equal $\hbar/k_B T = 8\pi GM/c^3$. From this analogy between a black body radiation and a black hole, one may conclude the radiation is indeed thermal and the temperature associated to the black hole is

$$T := \frac{\hbar c^3}{8\pi GM k_B} \quad (34)$$

This coincides with the classical definition of black hole temperature and simply performs that a radiating black hole is hot.

IX. WIDTH OF LINES

Having the above information, specially the probabilities (18) and (21), the mean value of emissive frequencies is easily evaluated,

$$\langle \omega \rangle = \frac{\eta \gamma \chi \omega_o}{C}, \quad (35)$$

where $\eta := \sum_{\text{all } \zeta} \frac{q(\zeta)}{(q(\zeta)-1)^2}$. A curious reader is encouraged to follow up the easy calculation in Appendix (E).

By the use of the probability of decays in (16), the mean value of the dimension of the emission sequences j is $\Delta t/\tau$. Thus, the mean decrease of the mass of black hole during the course of Δt is

$$\frac{\Delta \overline{M}}{\Delta t} = -\frac{\hbar \langle \omega \rangle}{c^2 \tau}. \quad (36)$$

On the other hand, the Stefan-Boltzmann law of black-body radiation from a black hole of horizon area A and surface temperature (34) indicates that the radiance rate from the black hole is

$$\frac{\Delta \overline{M}}{\Delta t} = -\frac{\hbar c^4}{15360\pi G^2 M^2}. \quad (37)$$

Comparing these two radiance rates of (37) and (36) we can evaluate τ ,

$$\tau = \frac{1920\pi \eta \gamma \chi}{C \omega_o}, \quad (38)$$

By definition in (16) τ is the survival time scale of the black hole from decaying. On average the time elapsed before a decay is

$$\bar{t} = \int_{t=0}^{\infty} t P_t(j=1) dt = 2\tau, \quad (39)$$

The uncertainty of the elapsing time before a decay is

$$(\Delta t)^2 = \int_{t=0}^{\infty} (t - \tau)^2 P_t(j=1) dt = 3\tau^2. \quad (40)$$

Due to the uncertainty principle $\Delta E \Delta t \geq \hbar/2$ and the definition of the frequency by energy, $E = \hbar \omega$, the uncertainty of the frequency turns out to be $\Delta \omega \simeq 1/\tau$. Therefore, the width of emission frequencies is proportional to $W = 1/\tau$,

$$W = \left(\frac{C}{1920\pi \eta \gamma \chi} \right) \omega_o, \quad (41)$$

To estimate the order of it, let us substitute the numerical values that are provided after $\gamma = \ln 3/\pi\sqrt{2}$. We apply the values of C and η from the Appendices (D) and (E). In $SU(2)$ representation, where $\chi = 0.5$, $\eta = 9.01$, and $C = 2$, the ratio becomes $W = 0.00029\omega_o$. In $SO(3)$ group representation, $\chi = \sqrt{2}$, $\eta = 4.7$, and $C = 0.93$ the ratio turns out to be $W = 0.00009\omega_o$. The order of lines width ratio is a few thousandth of the gap between the lines, thus the spectral lines are reasonably narrow.

X. THE SPECTRUM

In this section, the spectrum is reviewed.

Comparing the intensities corresponding to the frequencies $\omega_n(\zeta) = n\varpi(\zeta)$ and $\omega_m(\zeta') = m\varpi(\zeta')$, depending on whether the generations are the same or not, there exist two cases:

(i) In a generation, $\zeta = \zeta'$, the relative intensity of two harmonic frequencies is

$$\frac{I_n}{I_m} = \frac{n}{m} q(\zeta)^{m-n} \quad (42)$$

(ii) In different generations, $\zeta \neq \zeta'$, the relative intensities of the two modes $\omega_n(\zeta)$ and $\omega_{n'}(\zeta')$ is

$$\frac{I_n(\zeta)}{I_{n'}(\zeta')} = \frac{n}{n'} e^{-\Lambda[n\varpi(\zeta) - n'\varpi(\zeta')]} \quad (43)$$

Graphically, in Fig. (5) the intensities of harmonic frequencies corresponding to two different generations are shown in two different colors. The spectrum of harmonic frequencies corresponding to the fundamental frequency $\varpi(\zeta)$ is in black and the ones corresponding to $\varpi(\zeta')$ (for $\omega(\zeta') > \omega(\zeta)$) is in red (the thicker set of bar lines). The envelop of each generation matches with the one of Hawking and Bekenstein semiclassical result.

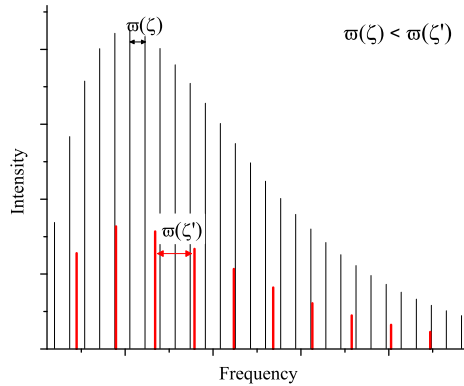


FIG. 5: The intensities of harmonic frequencies of two generations ζ and ζ' subject to the condition $\varpi(\zeta) < \varpi(\zeta')$.

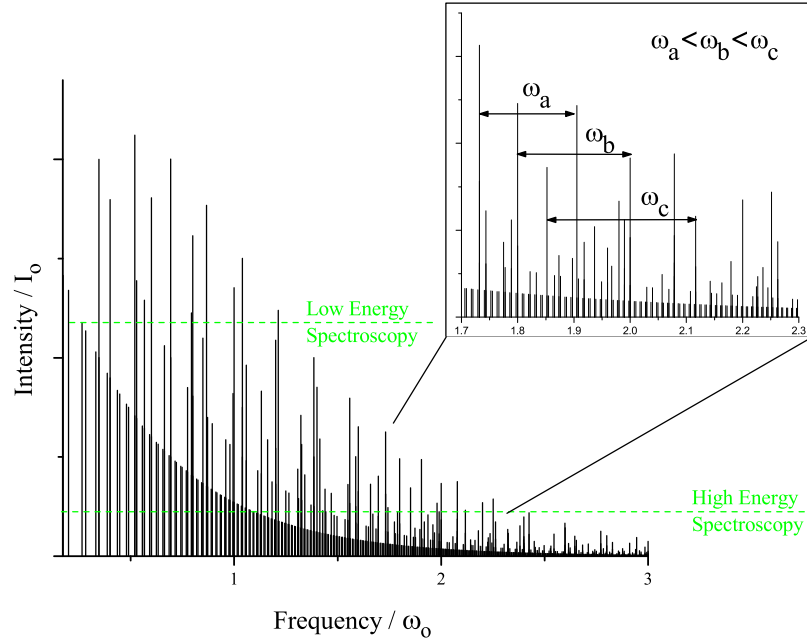


FIG. 6: The radiation spectrum of a loop quantum black hole.

Let us explain the disordered intensities by the following example. Consider three *consecutive* harmonic frequency modes ω_1, ω_2 and ω_3 where $\omega_i = n_i \varpi(\zeta_i)$ for $i = 1, 2, 3$ and $\omega_1 < \omega_2 < \omega_3$. Since n_1, n_2 , and n_3 are arbitrary integers in general, let us assume that the fundamental frequencies ϖ_1 and ϖ_3 , associated to the frequencies ω_1 and ω_3 respectively, are equal and the double of the fundamental frequency ϖ_2 associated to ω_2 ; ($\varpi_1 = \varpi_3$ and $\varpi_1 = 2\varpi_2$). Since there is no other line between these three lines, $n_3 = n_1 + 1$ and $n_2 = 2n_1 + 1$. Comparing the intensities associated to these three lines from (42), it turns out that the intensity of ω_2 is doubled, thus the middle line is *much* brighter than the two nearby ones.

Figure (6) shows the intensities corresponding to the harmonic frequencies up to the maximum $3\omega_0$. It is easy to see only a countable number of the most bright line exist in any interval of the order of ω_0 ; those which belong to the first few generations. In fact the intensity formula (42) shows the intensities corresponding to the third generation on are highly suppressed relative to the first two generations.

Let us recall that in addition to these lines, there are non-harmonic lines too but

since their intensities are extremely suppressed, they do not blend the discreteness of the most bright lines.

The maximum intensities in the spectrum belong to the frequencies of the condition $\omega_{\text{peak}} \sim 1/\Lambda = c^3/8\pi GM = \omega_o/\pi$. Therefore, the most bright lines are of the harmonics of integer valued number $n_{\text{peak}} \sim (\gamma\pi\chi\sqrt{\zeta})^{-1}$. Among all of the parameter, only the Barbero-Immirzi parameter is not certainly known and the dependency of the peak to the parameter is remarkable for the purpose of a possible way to determining it.

XI. DISCUSSION

The discreteness of area eigenvalues comes about the canonical quantization of 3-geometry because it is supposed that geometry has a distributional character with 1-dimensional excitations. Having this, the quantum geometrical operators are constructed by the canonical variables of loop quantum geometry. Among them, the area operator is the one whose corresponding eigenvalues are completely known.

In part (I) it was demonstrated that the area eigenvalues exhibits an unexpected symmetry. In fact, the spectrum of the numbers can be split into equidistant sequences of numbers. Each one of these evenly spaced sets of numbers is called a ‘generation.’ Each generation possesses an individual gap between levels, by which it is identified. The gap is proportional to the square root of a square-free number in $SO(3)$ representation, or the discriminant of a positive definite quadratic form in $SU(2)$ representation.

Consequently, the eigenvalues of area operator, instead of being labeled by three free numbers j_u, j_d and j_{u+d} , can be performed by fewer numbers; which can be the representative that specifies the generation and the level within a generation.

The relation between area and mass of a black hole (valid only on a black hole horizon), introduces quanta of energy by the use of the ‘area’ states of horizon. Having the symmetry of the quanta of horizon area, two different types of area transitions are possible: the transition either (i) between area levels within a generation (the o-called ‘generational transitions’) or (ii) between the area levels of different generations (the so-called ‘inter-generational transitions’). One of the immediate consequences of this symmetry is there appears a discrimination between these two types of transitions. Those quanta emitting from generational transitions can be reproduced

in many copies from many levels of a generation. However, there exists only one copy of each inter-generational transitions. This leads to a discrimination in the population of generational transitions motivated by the quantum amplification effect.

In Part (II) the intensity of radiation for any frequency was worked out. It was illustrated that a black hole radiates a continuous spectrum of frequencies. The spectrum of the quanta frequencies ranges from zero to a maximum. Nonetheless, there exist some spectral lines which take *additional* intensities due to the quantum amplification effect. This ‘amplification’ is a features of loop quantization of area. Following this, black hole radiation is dominated by the amplified area fluctuations and some discrete bright lines appear.

The smaller a fundamental frequency is, the more bright the harmonics are. Due to the γ -dependency of the intensity function and according to figure (6) the most bright lines in various energy scales of the spectrum belong to the first (or a few of the first) generation. Since the spectral lines are sufficiently narrow and apart from each other, they unlikely blend. In fact, the width of the lines are expected to be of the order of a thousandth of the frequency scale factor ω_o , while the gap between intensity peaks are of the order of this factor. Thus it is expected that such a quantum black hole radiates in a *visually discrete* pattern.

The precise spectroscopy depends on the exact value of the Barbero-Immirzi parameter as well as the group representation of spin network states.

Among the possible predictions of a canonically quantized black holes there are some features: 1) the radiation is effectively is visually discrete to observation, and 2) the intensities of consecutive lines are not orderly distributed.

Figure (7) is a typical expected radiance spectrum of a canonically quantized black hole is generated in low energy spectroscopy of figure (6). If the actual spectrum of black radiation is effectively discrete, the detection of a few of the most bright lines will be adequate to justify experimentally this prediction. The most bright lines in the spectrum belong to the first generation and the gap is of the order of the frequency scale $\omega_o \sim 10^{16}/M_{(\text{kg})}$ (eV).

Recently, a number of efforts have been put on the discovery of the radiance patterns of different types of black holes, the primordial holes [28, 29] and the one of higher dimensions [30]. One way to detect primordial black holes is by their Hawking radiation. The prediction of a canonically quantum black hole is also amenable to experimental check if the primordial black holes are founded. For instance, if the pri-

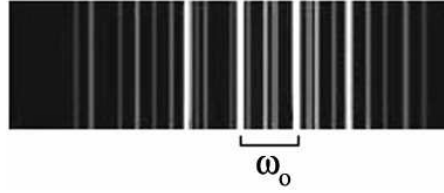


FIG. 7: A typical spectrum of a canonically quantized black hole radiation for the low energy spectroscopy of Fig (6).

mordial black holes constitute an essential part of dark matter in the galactic distant, the observation of a few of their most bright radiance lines can be within the modern sensitivity and can be possibly distinguished from the radiation of other objects.

It should also be noted that this radiation, if associated to the primordial black holes, is far beyond the Trans-Planckian problem of inflationary cosmology. The Trans-Planckian problem refers to the derivation of physical quantities from quantum field theory beyond the Planck scale. However, the proposed spectroscopy is based on a version of quantum gravity in which the difficulties within the semiclassical approximation does not exist.

Among important questions that are asked about the developments of area operator and black hole physics, there remain some important questions.

From experimental point view,

- One of the most incredibly important questions is that how far are we from detecting this spectrum?
- Considering a tiny percentage of dark matter obtaining from primordial black holes, is it possible to verify the spectrum as an alternative instead of gamma-ray busters?
- In the case of a rotating black holes, how the spectral lines are shifted or widen?

From numerical point of view,

- What is the correlation between the Barbero-Immirzi parameter (γ) and the visual spectrum at different energy spectroscopies? In other words, considering

the γ -dependency of the intensities, what are the visual frequencies in a low energy spectroscopy?

From theoretical point of view,

- Under imposing what conditions the quantum dynamics of black hole spin network states are identified?
- What is the Planck scale corrections to the entropy of such a quantum black hole?

It will be also interesting to see if a similar pattern can be illustrated for near extremal black holes in supergravity and string theories.

XII. ACKNOWLEDGEMENT

Helpful discussions with Lee Smolin are acknowledged.

Part III

Appendix

APPENDIX A: Area spectrum in $SO(3)$ version

Theorem 1: The set of numbers evaluated by the generating formula $\frac{1}{2}[2a(a+1) + 2b(b+1) - c(c+1)]$, where a, b , and c are positive integers and $c \in \{|a-b|, \dots, a+b-1, a+b\}$, is reduced into the whole \mathbb{Z}^+ , modulo rematching.

Proof: Suppose $a \geq b$. Let us consider the two independent numbers are $a = b + n$, where n is a positive integer. The subset $c = a + b$ is generated by the formula $n(n+1)/2 + b$. The first term is called Triangular numbers and are integers. The second term is independent from the first term and can be any positive integer.

This subset generates all positive integers and since other subsets generate integers, all of them fit into the whole positive integer sets \mathbb{Z}^+ . This set is a reduced set of the original one subject to identifying all repeated numbers.

The fundamental theorem of arithmetic states that every positive integer (except the number 1) can be represented in exactly one way as a product of one or more primes, apart from rearrangement. This theorem is also called ‘the unique factorization theorem’. Thus, prime numbers are the ‘basic building blocks’ of the natural numbers.¹¹ Decomposing any natural number into its prime numbers, the primes are either repeated or not. Collecting the natural numbers whose prime factors are not repetitive the square-free sequence of numbers are produced.

► **Definition of Square-free Numbers:** an integer number is said to be square-free, if its prime decomposition contains no repeated factors. For example, 30 is square free since its prime decomposition $2 \times 3 \times 5$ contains no repeated factors. In this note, this sequence is indicated by the symbol \mathbb{A} . Some of the known square-free numbers are given in table (I).

$$\mathbb{A} = \{1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35, 37, 38, 39, 41, 42, 43, 46, 47, 51, 53, 55, 57, 58, 59, 61, 62, 65, 66, 67, 69, 70, 71, 73, 74, 77, 78, 79, 82, 83, 85, 86, 87, 89, 91, 93, 94, 95, 97, 101, 102, 103, 105, 106, 107, 109, 110, 111, 113, \dots\}.$$

TABLE I: Square-free numbers (Sloane’s A005117)

There is no known polynomial algorithm for recognizing square-free, [31]. ◀

A natural number is the multiplication of a square-free number and a square number.

¹¹ In number theory, the prime factors of a number are considered as indistinguishable building blocks of numbers and thus the ordering of numbers does not matter.

Theorem: The natural numbers can be rewritten as a mixture of square generations by the contribution of all square-free representatives,

$$\{\mathbb{N}\} \equiv \bigcup_{\zeta \in \mathbb{A}} \{\zeta \mathbb{N}^2\}. \quad (\text{A1})$$

Proof: Any natural number can be written in terms of its prime factors, say $p_1^{n_1} \times p_2^{n_2} \cdots \times p_i^{n_i}$, where p_1, p_2, \dots , and p_i are different prime numbers and the exponents n_1, n_2, \dots, n_i are positive integers. These exponents are either even or odd numbers. In the most general case all of the exponents are different odd numbers, $n_i = 2m_i + 1$. Therefore, the above-mentioned number can be rewritten in the form $(p_1 \times p_2 \cdots \times p_i) \times (p_1^{m_1} \times p_2^{m_2} \cdots \times p_i^{m_i})^2$. Due to the assumption that the prime number p 's are different, the first parenthesis is equivalent to a square-free number and the second parenthesis is nothing but n^2 for the natural number $n = p_1^{m_1} \times p_2^{m_2} \cdots \times p_i^{m_i}$. Therefore, $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}$ and $a \in \mathbb{A}, x \equiv a \times y^2$.

In Table II, having the first 15 square-free numbers of \mathbb{A} , the corresponding elements of the square generations $\zeta \mathbb{N}^2$ are tabulated up to the first five elements.

	$1\mathbb{N}^2$	$2\mathbb{N}^2$	$3\mathbb{N}^2$	$5\mathbb{N}^2$	$6\mathbb{N}^2$	$7\mathbb{N}^2$	$10\mathbb{N}^2$	$11\mathbb{N}^2$	$13\mathbb{N}^2$	$14\mathbb{N}^2$	$15\mathbb{N}^2$	$17\mathbb{N}^2$	$19\mathbb{N}^2$	$21\mathbb{N}^2$
$m=1^a$	1	2	3	5	6	7	10	11	13	14	15	17	19	21
$m=2$	4	8	12	20	24	28	40	44	52	56	60	68	76	84
$m=3$	9	18	27	45	54	63	90	99	117	126	135	153	171	189
$m=4$	16	32	48	80	96	112	160	176	208	224	240	272	304	336
$m=5$	25	50	75	125	150	175	250	275	325	350	375	425	475	525

^aThis row shows the representatives ζ .

TABLE II: The first fifteen elements of some $SO(3)$ based generations $\zeta \mathbb{N}^2$.

A column in the Table (II) indicates the elements of a square generation and consists of all natural number up to 21. By extending this Table, the consistency of the elements with natural numbers can be verified up to any order. There is no common element in different square generations,

$$\forall \zeta_1, \zeta_2 \in \mathbb{A}, \text{ if } \zeta_1 \neq \zeta_2, \quad \{\zeta_1 \mathbb{N}^2\} \cap \{\zeta_2 \mathbb{N}^2\} = \emptyset. \quad (\text{A2})$$

Consider the sequence of numbers that contain the same square-free ζ factor multiplied by all square numbers, $\zeta\mathbb{N}^2$. Taking square root from the elements of the square generation the *equidistant* sequence $\sqrt{\zeta}\mathbb{N}$ is produced.

APPENDIX B: Area spectrum in $SU(2)$ version

Theorem: The set of numbers evaluated by the generating formula $4[2a(a+1) + 2b(b+1) - c(c+1)]$, where a and b are positive integer or half-integer of $\frac{1}{2}\mathbb{Z}$ and $c \in \{|a-b|, \dots, a+b-1, a+b\}$, is reduced into the congruent number unto 0 and 3 mode 4 modulo the degeneracies.

Proof: Suppose $a \geq b$. Let us consider the two independent numbers are $a = b + n$, where $n \in \frac{1}{2}\mathbb{Z}$. The subset $c = a + b$ is generated by the formula $4n(n+1) + 8b$. Let us consider $n = N/2$ and $b = B/2$ where N and B are independent natural numbers. Substituting them in the formula it becomes $N(N+2) + 4B$. The first term is the mixture of congruent numbers unto 0 or 3 mod 4. The second term is the congruent numbers unto 0 mod 4. In other words, a number that is generated by $N(N+2)$ is either $4m+3$ or $4m$ for some integers m . This is not changed when the term $4B$ is added to the numbers. Let us fix N unto either 0 or 1. The whole sequence of congruent numbers unto 0 and 3 mod 4 are obviously generated from $4B+3$ and $4B$ for any integer B . All other numbers fit to the whole sequence if one identifies all degeneracies.

Evaluating $4(m_{j_u, j_d, j_u+d})^2$ in $SU(2)$ group representation, the Skew Amenable numbers are produced.

► **Definition of Skew Amenable numbers:** in a simple definition these numbers are the page numbers of a book that is printed out only at the pages that come after each two leaves of sheets by face. This can be interpreted in a mathematical language as the congruent numbers to either 0 or 3 in mod 4.¹²

¹² There is also another definition that a number n is skew amenable if there exist a set of integers $\{m_i\}$ satisfying the relations: $n = \sum_{i=1}^n m_i = -\prod_{i=1}^n m_i$, [26]. For instance, the number 8 is a skew amenable because it can be decomposed into an 8 term sum as well as the negated product of exactly the same numbers: $8 = 1 + 1 + 1 + 1 + 1 + 1 - 2 + 4 = -(1 \times 1 \times 1 \times 1 \times 1 \times 1 \times (-2) \times 4)$. Another example is 3, which satisfied the condition: $3 = 1 + 3 - 1 = -(1 \times 3 \times (-1))$.

The Skew Amenable numbers smaller than 200 are 3, 4, 7, 8, 11, 12, 15, 16, 19, 20, 23, 24, 27, 28, 31, 32, 35, 36, 39, 40, 43, 44, 47, 48, 51, 52, 55, 56, 59, 60, 63, 64, 67, 68, 71, 72, 75, 76, 79, 80, 83, 84, 87, 88, 91, 92, 95, 96, 99, 100, 103, 104, 107, 108, 111, 112, 115, 116, 119, 120, 123, 124, 127, 128, 131, 132, 135, 136, 139, 140, 143, 144, 147, 148, 151, 152, 155, 156, 159, 160, 163, 164, 167, 168, 171, 172, 175, 176, 179, 180, 183, 184, 187, 188, 191, 192, 195, 196, 199, 200, etc. (Sloane's A014601)¹³ ◀

It is known in number theory that any square integer number is the congruent to either 0 or 1 mod 4. On the other hand, there is a theorem that in the existence of two equalities $x_1 \equiv x_2 \pmod{m}$ and $x_3 \equiv x_4 \pmod{m}$ of the same modular m , it can be it is easy to verify that $x_1 \times x_3 \equiv x_2 \times x_4 \pmod{m}$. Accordingly, having one of the two equality as $x_1 = 0$ or 3 (mod4) for a Skew Amenable number x_1 , and the equality $x_2 = 0$ or 1 (mod4) for a square number x_2 , the multiplication of these two produces a Skew Amenable number $x_1 \times x_2 = 0$ or 3 (mod4) is generated. In other words, multiplying the complete set of Skew Amenable numbers and the complete set of square numbers, the product a subset of the Skew Amenable numbers is generated.

Having this fact in mind, for any random Skew Amenable number b' there exists a corresponding Skew Amenable number b that satisfies the equality $b' = b \times n^2$ for $n \in \mathbb{N}$. In fact, the set of numbers b for all $n \in \mathbb{N}$ is a subset of Skew Amenable numbers. We represent this subset by the symbol \mathbb{B} . Now the question is: what is \mathbb{B} ? To answer the questions, the Skew Amenable number can be generated and the elements of the set \mathbb{B} are identified individually. The result is in the table (III).

$$\mathbb{B} = \{3, 4, 7, 8, 11, 15, 19, 20, 23, 24, 31, 35, 39, 40, 43, 47, 51, 52, 55, 56, 59, 67, 68, 71, 79, 83, 84, 87, 88, 91, 95, 103, 104, 107, 111, 115, 116, 119, 120, 123, 127, 131, 132, 136, 139, 143, 148, 151, 152, 155, 159, 163, 164, 167, 168, 179, 183, 184, 187, 191, \dots\}.$$

TABLE III: The Discriminants of the Positive Definite Quadratic Forms (Sloane's A003657)

¹³ <http://www.research.att.com/~njas/sequences/A014601>

The negative of this sequence of numbers coincide with a well-known sequence of discriminants of *the Positive Definite Quadratic Forms*, [32]. The definition of the positive definite quadratic forms is explained in the Appendix (C) of this note.

Consequently, an Skew Amenable sequence of number, which is generated from the evaluation of $4(m_{j_u, j_d, j_{u+d}})^2$ in $SU(2)$ representation, can be rewritten as an element of the set \mathbb{B} multiplied by an integer squared. In other words, the elements of the set $4(m_{j_u, j_d, j_{u+d}})^2$ can be represented as square generations with representative elements of the set \mathbb{B} . In table IV sixteen elements of the square generations whose representatives are the first five elements of the set \mathbb{B} , is tabulated.

	$3\mathbb{N}^2$	$4\mathbb{N}^2$	$7\mathbb{N}^2$	$8\mathbb{N}^2$	$11\mathbb{N}^2$	$15\mathbb{N}^2$	$19\mathbb{N}^2$	$20\mathbb{N}^2$	$23\mathbb{N}^2$	$24\mathbb{N}^2$	$31\mathbb{N}^2$	$35\mathbb{N}^2$	$39\mathbb{N}^2$	$40\mathbb{N}^2$	$43\mathbb{N}^2$
m=1 ^a	3	4	7	8	11	15	19	20	23	24	31	35	39	40	43
m=2	12	16	28	32	44	60	76	80	92	96	124	140	156	160	172
m=3	27	36	63	72	99	135	171	180	207	216	279	315	351	360	378
m=4	48	64	112	176	240	304	320	368	384	384	496	560	624	640	688
m=5	75	100	175	200	275	375	475	500	575	600	775	875	975	1000	1075

^aThis row shows the representatives ζ .

TABLE IV: The first sixteen elements of some $SU(2)$ based generations $\zeta\mathbb{N}^2$.

In the table (IV), all elements of the Skew Amenable numbers up to 44 are present and any extension of the table will produce all of the others up to any order.

There is no common element in different square generations,

$$\forall \zeta_1, \zeta_2 \in \mathbb{B}, \text{ if } \zeta_1 \neq \zeta_2, \quad \{\zeta_1\mathbb{N}^2\} \cap \{\zeta_2\mathbb{N}^2\} = \emptyset. \quad (\text{B1})$$

APPENDIX C: Positive definite quadratic forms

A quadratic form is a two-variable integer-valued function $f(x, y) = ax^2 + bxy + cy^2$, with $a, b, c \in \mathbb{Z}$. This form 'primitive' if a, b, c are relatively 'prime'. The 'discriminant' of this form is defined as $\Delta := b^2 - 4ac$.

Substituting integers values in two variables x and y respectively, the form is evaluated by an integer m , $f(x_0, y_0) = m$. This problem can be restated as follows, the

‘representation’ of the form $f(a, b, c) = m$ is elements of the solution space of the equation (x_0, y_0) . A representation is called ‘primitive’ if $\gcd(x_0, y_0)=1$.

Given a form f , the transformation $x = \alpha x' + \beta y'$ and $y = \gamma x' + \delta y'$ transforms f into f' . The new form f' remains as an integer if and only if $\alpha\delta - \beta\gamma = \pm 1$. The interesting fact is that in such a transformation that preserves the integer character of the form the discriminant Δ remains *invariant*.

If f is a form of integer m , we can rewrite the definition of a form as $4am = (2ax + by)^2 - \Delta y^2$. In the case that Δ is a perfect square number the right hand side is written in the form $(2ax + by + y\sqrt{\Delta})(2ax + by - y\sqrt{\Delta})$. In this case the different representations of solutions are *degenerate* and thus indistinguishable. These forms are of this note.

In the case of $\Delta < 0$, it is clear from $4am = (2ax + by)^2 - \Delta y^2$ that for any representation (x, y) , m and a (and c) are of the same sign. A forms whose corresponding discriminant is negative is called a ‘definite form’ and if m is positive, it is called a ‘positive definite form’.¹⁴

By substituting the positive a and c in $f(x, y) = ax^2 + bxy + cy^2$, if the form for any representation (x, y) become positive and the discriminant becomes negative, the form is a definite positive quadratic form. Evaluating the negated values of the discriminants of such forms produces the following sequence of numbers 3, 4, 7, 8, 11, 15, 19, 20, 23, 24, 31, 35, 39, 40, 43, 47, 51, 52, 55, 56, 59, 67, 68, 71, 79, 83, 84, 87, 88, 91, 95, 103, 104, 107, 111, 115, 116, 119, 120, 123, 127, 131, 132, 136, 139, 143, 148, 151, 152, 155, 159, 163, 164, 167, 168, 179, 183, 184, 187, 191, \dots , which is the same elements of the set \mathbb{B} in table (III).

For instance, $(a = 1, b = 1, c = 1)$ defines a positive definite form whose corresponding discriminant is -3 and its negated value is the first element of \mathbb{B} . The second element is produced after $(a = 1, b = 0, c = 0)$.

To verify that this sequence is the congruent to either 0 or 3 mod4, it is enough check the consistency of the negated discriminant $-\Delta = 4ac - b^2$ with this congruent. It is clear that b^2 is congruent to 0 or 1 (mod4). Also, $4ac$ is the congruent to 0 mod4. Therefore, $-\Delta$ is congruent to 0 or 3 (mod4).

¹⁴ In the case that $\Delta > 0$, a and c are of opposite signs and thus both positive an negative integers m may be represented on f . This case is called indefinite form and is not of our interest of study.

APPENDIX D: The normalization coefficient C

To calculate this coefficient, it should be noticed the spectrum of non-harmonics is almost continuous except at zero and the harmonics, $\omega' \in \mathbb{R}^+ - \{0\} - \{\text{harmonics}\}$. These frequencies are all uniformly weighted by $\rho = 1/N_o$. Since the population of harmonics is much more than the non-harmonics, we can approximate the population of a harmonics to be $N - 1$ instead of N and add the one copy of each harmonic into the above mentioned set of frequencies in order to fill the gaps. Doing so, the equally weighted set of frequencies $\omega' \in \mathbb{R}^+ - \{0\}$ is provided. By the use of (18) and (21), in the classical limits ($N_o \gg 1$), the normalization coefficient reads

$$\begin{aligned} C &= \lim_{N \rightarrow \infty} \sum_{\text{all } \zeta} \sum_{n=1}^N \frac{N-1}{N} \rho(\zeta) q(\zeta)^{-n} + \int \frac{1}{N_o} e^{-\Lambda \omega'} \Lambda d\omega' \\ &\simeq \sum_{\text{all } \zeta} \sum_{n=1}^{\infty} \rho(\zeta) q(\zeta)^{-n}. \end{aligned} \quad (\text{D1})$$

By the use of the algebraic relation $\sum_{n=1}^{\infty} x^{-n} = 1/(x-1)$, the internal sum in C is summarized. Consequently, the normalization coefficient becomes

$$C = \sum_{\text{all } \zeta} \frac{\rho(\zeta)}{q(\zeta) - 1} \quad (\text{D2})$$

It is useful to check the finiteness of the normalization coefficient C .

The finiteness of C : for the purpose of simplicity the definition $q(\zeta) = \exp(\pi\gamma\chi\sqrt{\zeta})$ can be rewritten $q(\zeta) := h^{\sqrt{\zeta}}$, where $h := \exp(\pi\gamma\chi)$ is in both groups greater than 1.

The Cauchy root method of convergence test is such that for series like $\sum_n a_n$, if the value of $\lim_{n \rightarrow \infty} |a_n|^{1/n}$ is smaller than one, the series converges. In the series sum of C , this condition reads

$$\lim_{\sqrt{\zeta} \rightarrow \infty} \left| \frac{1}{\sqrt{\zeta}} \frac{1}{h^{\sqrt{\zeta}} - 1} \right|^{\frac{1}{\sqrt{\zeta}}} \sim \lim_{\sqrt{\zeta} \rightarrow \infty} \left(\frac{h^{-\sqrt{\zeta}}}{\sqrt{\zeta}} \right)^{\frac{1}{\sqrt{\zeta}}} = \frac{1}{h} < 1. \quad (\text{D3})$$

Therefore C is a finite number.

Numerical work can estimate the range of C . Substituting ρ from (23) and Barbero-Immirzi parameter from [10], $\gamma = \ln 3 / \pi \sqrt{2}$, in $q(\zeta)$ the coefficient C is simplified to $\sum_{\zeta} \zeta^{-1/2} (3^{\chi \sqrt{\zeta/2}} - 1)$. In the $SU(2)$ representation, where $\zeta \in \mathbb{B}$ and $\chi = 1/2$, turn out to be $C = 2.01$, whilst in the $SO(3)$ representation group, where $\zeta \in \mathbb{A}$ and $\chi = \sqrt{2}$, it is $C = 0.93$.

APPENDIX E: Average of frequency $\langle \omega \rangle$

Since the probabilities of (18) and (21) are normalized, the mean value of the emitting frequencies is

$$\langle \omega \rangle := \sum_{\zeta} \sum_{n=1}^{\infty} \omega_n(\zeta) P_{\Delta t}(\{\omega_n(\zeta)\} | 1) + \left(\frac{1}{N_o} \right) \int_0^{\infty} \omega e^{-\Lambda \omega} \Lambda d\omega.$$

The second term for a classical black hole is negligible comparing to the first term. Using (21) and the algebraic formula $\sum_{n=1}^{\infty} n x^n = x/(1-x)^2$, where $x < 1$, the mean value of the frequency of a generation ζ turns out to be $(1/C) \varpi(\zeta) \rho(\zeta) q(\zeta) / (q(\zeta) - 1)^2$ and therefore the mean value of all frequencies becomes

$$\langle \omega \rangle \sim \frac{1}{C} \sum_{\zeta} \varpi(\zeta) \frac{\rho(\zeta) q(\zeta)}{(q(\zeta) - 1)^2} \quad (\text{E1})$$

Using (13) and (23), we can rewrite the equation in the form

$$\langle \omega \rangle \sim \frac{\omega_o \gamma \chi}{C} \sum_{\zeta} q(\zeta) / (q(\zeta) - 1)^2. \quad (\text{E2})$$

Convergence of $\langle \omega \rangle$: To check the convergence of the sequence $\sum_{\zeta} q(\zeta) / (q(\zeta) - 1)^2$ via the Cauchy's convergence test, since the real free index in the function $q(\zeta)$ is $\sqrt{\zeta}$ the test function $\lim_{\sqrt{\zeta} \rightarrow \infty} |q(\zeta) / (q(\zeta) - 1)^2|^{1/\sqrt{\zeta}}$ should be considered. By the use of the definition of $q(\zeta) := h^{\sqrt{\zeta}}$, where $h := \exp(\pi \chi \gamma) > 1$,

$$\lim_{\sqrt{\zeta} \rightarrow \infty} \left| \frac{1}{\sqrt{\zeta}} \frac{h^{\sqrt{\zeta}}}{(h^{\sqrt{\zeta}} - 1)^2} \right|^{\frac{1}{\sqrt{\zeta}}} \sim \frac{1}{h} < 1. \quad (\text{E3})$$

This summation converges.

Let us rewrite the sum by the use of the definition $\eta := \sum_{\zeta} q(\zeta)/(q(\zeta) - 1)^2$ as $\langle \omega \rangle = \eta \gamma \chi \omega_o / C$. Having the parameter from Appendix (D), and by substituting $\gamma = \ln 3 / \pi \sqrt{2}$ the coefficient η in the $SU(2)$ representation turns out to be $\eta = 9.0$, while it is $\eta = 1.7$ in the $SO(3)$ group representation. By the use of the numerical values of C , the mean value of frequency $\langle \omega \rangle$ is either about $11\omega_o$ in $SU(2)$ group or about $15\omega_o$ in $SO(3)$ group.

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